

# WILD RAMIFICATION OF NILPOTENT COVERINGS AND COVERINGS OF BOUNDED DEGREE

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**ABSTRACT.** A finite étale map between irreducible, normal varieties is called tame, if it is tamely ramified with respect to all partial compactifications whose boundary is the support of a strict normal crossings divisor. We prove that if the Galois group of a Galois covering contains a normal nilpotent subgroup of index bounded by a constant  $N$ , then the covering is tame if and only if it is tamely ramified with respect to a single distinguished partial compactification only depending on  $N$ . The main tools used in the proof are Temkin's local purely inseparable uniformization and a Lefschetz type theorem due to Drinfeld.

## 1. INTRODUCTION

Let  $k$  be a field and let  $X$  be a separated, finite type  $k$ -scheme. Following [KS10], a finite étale covering  $f : Y \rightarrow X$  is called *tame* if for all  $k$ -morphisms  $\varphi : C \rightarrow X$ , with  $C$  a regular  $k$ -curve, the induced covering  $\varphi_C : Y \times_X C \rightarrow C$  is tamely ramified with respect to  $\overline{C} \setminus C$ , where  $\overline{C}$  is the unique regular proper curve with function field  $k(C)$ . Thus to check whether a given covering  $Y \rightarrow X$  is tame, a priori, one has to test tameness along infinitely many maps of curves to  $X$ .

If  $X$  is regular and connected, it is known ([KS10, Thm. 4.4]) that  $f : Y \rightarrow X$  is tame if and only if  $f$  is tamely ramified with respect to every geometric discrete valuation on  $k(X)$ . Equivalently,  $f$  is tame if and only if it is tamely ramified with respect to every good partial compactification  $(X, \overline{X})$  (Definition 7), i.e., if the normalization  $\bar{f} : \overline{Y} \rightarrow \overline{X}$  of  $\overline{X}$  in  $Y$  is tamely ramified along the normal crossings divisor  $(\overline{X} \setminus X)_{\text{red}}$ . Again, to check whether a given covering is tame or not, one a priori has to check tameness with respect to infinitely many good partial compactifications.

However, if there exists a good compactification  $(X, \overline{X}_0)$ , that is, a good partial compactification with  $\overline{X}_0$  *proper*, then  $f$  is tame if and only if  $f$  is tamely ramified with respect to this single good compactification. Of course, as resolution of singularities is not known to be possible in positive characteristic, it is not known whether every regular  $k$ -variety  $X$  admits a good compactification.

The goal of this short note is to prove that for a certain class of coverings, tameness can be detected by a single, privileged good *partial* compactification.

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**Theorem 1.** Assume that the field  $k$  is either algebraically closed or finite. Let  $X$  be a smooth, geometrically irreducible, quasi-projective  $k$ -scheme and  $N \in \mathbb{N}$ . There exists a good partial compactification  $(X, \overline{X}_N)$  ([Definition 7](#)), such that for any Galois étale covering  $f : Y \rightarrow X$  with group  $G$ , such that  $G$  has a normal nilpotent subgroup of index  $\leq N$ , the following are equivalent:

- (a)  $f$  is tame ([Definition 9, \(c\)](#)),
- (b)  $f$  is tamely ramified with respect to  $(X, \overline{X}_N)$ .

□

The proof of the theorem uses Temkin's inseparable local uniformization ([\[Tem13\]](#)), following an idea of Kerz-Schmidt, together with a generalization of a Lefschetz theorem due to Drinfeld ([\[EK15, Prop. 6.2\]](#)).

**Example 2.** One particularly interesting class of Galois coverings to which [Theorem 1](#) applies arises as follows. Fix  $r \in \mathbb{N}$ . According to Jordan's theorem ([\[LP11, Thm. 0.1\]](#)) there exists a constant  $N(r)$ , only depending on  $r$ , such that any finite subgroup of  $\mathrm{GL}_r(\mathbb{C})$  contains a normal abelian subgroup of index  $\leq N(r)$ . Thus, the tameness of representations  $\rho : \pi_1(X, \bar{x}) \rightarrow \mathrm{GL}_r(\mathbb{C})$  with finite image can be detected on the good partial compactification  $(X, \overline{X}_{N(r)})$ . □

The structure of the article is as follows. In [Section 2](#) we gather some facts about discrete valuations and good partial compactifications, and in [Section 3](#) we present the necessary background from ramification theory. The proof of [Theorem 1](#) for nilpotent coverings is carried out in [Section 4](#), and the general case is completed in [Section 5](#).

## 2. GEOMETRIC DISCRETE VALUATIONS

In this section,  $k$  is an arbitrary field and  $X$  denotes a normal, irreducible, separated  $k$ -scheme of finite type.

**Definition 3.** Let  $k(X)$  denote the function field of  $X$ .

- (a) By a *discrete valuation* on  $k(X)$ , we mean a discrete valuation of rank 1, i.e. with value group isomorphic to  $\mathbb{Z}$ .
- (b) Let  $v$  be a discrete valuation on  $k(X)$ . We write  $\mathcal{O}_v \subseteq k(X)$  for its valuation ring,  $\mathfrak{m}_v$  for the maximal ideal of  $\mathcal{O}_v$  and  $k(v)$  for its residue field.
- (c) A *model* of  $k(X)$  is separated, integral, finite type  $k$ -scheme  $X'$  with  $k(X') = k(X)$ .
- (d) If  $X'$  is a model of  $k(X)$  and  $v$  a discrete valuation on  $k(X)$ , then a point  $x \in X'$  is called *center of  $v$* , if  $\mathcal{O}_{X',x} \subseteq \mathcal{O}_v \subseteq k(X)$ , and  $\mathfrak{m}_v \cap \mathcal{O}_{X',x} = \mathfrak{m}_x$ .
- (e)  $v$  is called *geometric* or *divisorial* or *of the first kind* if there exists a model  $X'$  of  $k(X)$  such that  $v$  has a center which is a codimension 1 point.

□

**Remark 4.** Recall that if  $X'$  is separated over  $k$ , then  $v$  has at most one center on  $X'$ , and if  $X'$  is proper, then  $v$  has precisely one center on  $X'$ .  $\square$

The question of whether a discrete valuation is geometric or not is encoded in the *Abhyankar inequality*:

**Proposition 5** ([Liu02, Ch. 8, Thm. 3.26]). *If  $v$  is a discrete valuation on  $k(X)$  with center  $x$  on  $X$ , then we have the inequality*

$$\mathrm{trdeg}_{k(x)} k(v) \leq \dim \mathcal{O}_{X,x} - 1. \quad (1)$$

*The discrete valuation  $v$  is geometric if and only if equality holds in (1).*  $\square$

From this one derives the following fact.

**Proposition 6** ([Liu02, Ch. 8, Ex. 3.16]). *If  $f : Y \rightarrow X$  is a dominant morphism of normal, irreducible, separated, finite type  $k$ -schemes, and if  $v$  is a geometric discrete valuation of  $k(Y)$ , then either  $\mathcal{O}_v \supseteq k(X)$ , or  $w := v|_{k(X)}$  is a geometric discrete valuation of  $k(X)$  with finitely generated residue extension  $k(w)/k(x)$ .*  $\square$

One can translate from the language of geometric discrete valuations into the language of good partial compactifications.

**Definition 7.** Let  $k$  be an arbitrary field and let  $X$  be a regular, irreducible, separated  $k$ -scheme of finite type. Consider a regular, separated  $k$ -scheme  $\overline{X}$  of finite type, together with a dominant open immersion  $X \hookrightarrow \overline{X}$ , such that the complement  $\overline{X} \setminus X$  is the support of a strict normal crossings divisor ([GM71, §1.8]). The pair  $(X, \overline{X})$  is said to be a *good partial compactification* of  $X$ . If  $\overline{X}$  is proper, then  $(X, \overline{X})$  is said to be a *good compactification* of  $X$ .  $\square$

Given a proper normal model  $X'_0$  of  $k(X)$  and a geometric discrete valuation  $v$  on  $k(X)$ , for  $n > 0$  let  $X'_n$  be the blow-up of  $X'_{n-1}$  in the center of  $v$ . According [Liu02, Ch. 8, Ex. 3.14], this process terminates and one obtains a model  $X''$  on which  $v$  has a codimension 1 center. This implies the following proposition.

**Proposition 8.** *Let  $X$  and  $k$  be as in Definition 7. If  $v = \{v_1, \dots, v_n\}$  is a set of geometric discrete valuations on  $k(X)$  which do not have a center on  $X$ , then there exists a good partial compactification  $(X, \overline{X}_v)$  of  $X$ , such that the codimension 1 points on  $\overline{X}_v \setminus X$  precisely correspond to  $\{v_1, \dots, v_n\}$ .*  $\square$

### 3. TAME RAMIFICATION

We continue to denote by  $k$  a field.

**Definition 9.** Let  $X$  and  $Y$  be normal, irreducible, separated, finite type  $k$ -schemes and  $f : Y \rightarrow X$  a finite étale  $k$ -morphism.

- (a) If  $v$  is a discrete valuation on  $k(X)$ , there exist finitely many discrete valuations  $w_1, \dots, w_r$  on  $k(Y)$  such that  $\mathcal{O}_{w_i} \cap k(X) = \mathcal{O}_v$ . Then  $f$  is *tamely ramified with respect to  $v$*  if the residue extensions  $k(w_i)/k(v)$  are separable and if the ramification indices on  $\mathcal{O}_v \subseteq \mathcal{O}_{w_i}$  are prime to  $\text{char}(k)$ .

Note that if  $v$  has center on  $X$ , then  $f$  is tame with respect to  $v$ , as  $f$  is étale.

- (b) If  $(X, \overline{X})$  is a good partial compactification of  $X$  (Definition 7), then  $f$  is said to be *tamely ramified with respect to  $(X, \overline{X})$*  if  $f$  is tamely ramified with respect to all discrete valuations on  $k(X)$  which are centered in codimension 1 points of  $\overline{X}$ . Or, equivalently, if  $f$  is tamely ramified with respect to the strict normal crossings divisor  $(\overline{X} \setminus X)_{\text{red}}$  ([GM71, §2]).

If  $\bar{x}$  is a geometric point of  $X$ , we write  $\pi_1^{\text{tame}}((X, \overline{X}), \bar{x})$  for the quotient of  $\pi_1^{\text{ét}}(X, \bar{x})$  corresponding to the category of finite étale coverings which are tamely ramified with respect to  $(X, \overline{X})$ . Concretely,  $\pi_1^{\text{tame}}((X, \overline{X}), \bar{x})$  is the quotient of  $\pi_1^{\text{ét}}(X, \bar{x})$  by the smallest closed normal subgroup containing all wild inertia groups attached to points of the normalization of  $\overline{X}$  in  $Y$  lying over the codimension 1 points of  $\overline{X} \setminus X$ .

- (c) The morphism  $f$  is called *tame* if for every regular  $k$ -curve  $C$  and for every  $k$ -morphism  $\varphi : C \rightarrow X$ , the induced étale covering

$$f_C : Y \times_X C \rightarrow C$$

is tamely ramified with respect to the good compactification  $(C, \overline{C})$ , where  $\overline{C}$  is the unique normal proper curve with function field  $k(C)$ .

If  $\bar{x}$  is a geometric point of  $X$ , we denote by  $\pi_1^{\text{tame}}(X, \bar{x})$  the quotient of  $\pi_1(X, \bar{x})$  corresponding to the category of tame étale coverings of  $X$  ([KS10, Section 7]).

- (d) As is customary, we will often use the word “wild” in place of “not tame”.

□

**Remark 10.** In [KS10, Rem. 4.5] it is proved that if an étale covering  $f$  as above is tame, then  $f$  is tamely ramified with respect to all geometric discrete valuations  $v$  on  $k(X)$ , or equivalently with respect to all good partial compactifications of  $X$ . Moreover, if  $X$  is regular, then [KS10, Thm. 4.4] shows that the converse is also true.

In particular, if  $\bar{x}$  is a geometric point of  $X$  and if  $(X, \overline{X})$  is a good partial compactification of  $X$ , then there is a canonical quotient map  $\pi_1^{\text{tame}}((X, \overline{X}), \bar{x}) \rightarrow \pi_1^{\text{tame}}(X, \bar{x})$ , which is an isomorphism if  $\overline{X}$  is proper. □

In the proof of Theorem 1 we will use the following two lemmas for which we could not find a reference.

**Lemma 11.** *Let  $k$  be a perfect field and fix an algebraic closure  $\bar{k}$  of  $k$ . Let  $X$  be a geometrically irreducible, regular, separated, finite type  $k$ -scheme. If  $\bar{x}$  is a geometric point of  $X \times_k \bar{k}$ , then there is a short exact sequence*

$$1 \longrightarrow \pi_1^{\text{tame}}(X \times_k \bar{k}, \bar{x}) \longrightarrow \pi_1^{\text{tame}}(X, \bar{x}) \longrightarrow \text{Gal}(\bar{k}/k) \longrightarrow 1$$

□

*Proof.* The argument is completely analogous to [SGA1, XI.6.1]. Inspection of this reference shows that the only thing to be checked is that if  $f : Y \rightarrow X_{\bar{k}}$  is a tame Galois covering, then  $f$  is the base change of a tame covering of  $X_{k'}$  for some finite extension  $k'/k$ . It is clear that  $f$  comes via base change from a Galois covering  $Z \rightarrow X_{k'}$  for some  $k'$ . Let  $\varphi : C \rightarrow X_{k'}$  be a nonconstant map from a regular  $k$ -curve  $C$  to  $X_{k'}$ . By enlarging  $k'$  we may assume that the points of  $\bar{C} \setminus C$  are  $k'$ -rational, where  $\bar{C}$  is the unique normal proper  $k'$ -curve compactifying  $C$ , and similarly for the points at infinity of  $Z \times_{X_{k'}} C$  lying over  $\bar{C} \setminus C$ .

Since a finite, totally ramified map of discrete valuation rings is tamely ramified if and only if the corresponding map on strict henselizations is tamely ramified, it follows that  $\varphi_Z : C \times_{X_{k'}} Z \rightarrow C$  is tame if and only if its base change to  $\bar{k}$  is tamely ramified.

Consequently,  $Z \rightarrow X_{k'}$  is tame, which is what we set out to prove. ■

**Lemma 12.** *Let  $K$  be a field equipped with a discrete valuation  $v$ ,  $L/K$  a finite separable extension and  $K'/K$  a finite purely inseparable extension. Let  $v'$  denote the unique extension of  $v$  to  $K'$ . Then  $v$  is tamely ramified in  $L/K$  if and only if  $v'$  is tamely ramified in  $L \otimes_K K'/K'$ .* □

*Proof.* First note that  $L' := L \otimes_K K'$  really is a field, as  $K'$  and  $L$  are linearly disjoint when embedded in any algebraic closure of  $K$ . Without loss of generality we may assume that  $K' = K[X]/(X^p - a)$  with  $a \in K \setminus K^p$ . Let  $w$  be a valuation on  $L$  extending  $v$  and  $w'$  the unique extension of  $w$  to  $L'$ . Note that the completion  $K'_v$  is either  $K_v$  or  $K_v[X]/(X^p - a)$ , depending on whether  $a$  is a  $p$ -th power in  $K_v$  or not. As  $L_w/K_v$  is separable,  $a$  is a  $p$ -th power in  $L_w$  if and only if it is a  $p$ -th power in  $K_v$ . Thus in the commutative diagram

$$\begin{array}{ccc} L_w & \longrightarrow & L'_{w'} \\ \uparrow & & \uparrow \\ K_v & \longrightarrow & K'_{v'} \end{array}$$

the horizontal arrows are either both purely inseparable of degree  $p$  or both isomorphisms. In either case  $L'_{w'} = L_w \otimes_{K_v} K'_{v'}$ . Write  $e$  and  $e'$  for the ramification indices of  $w|v$  and  $w'|v'$ , and  $f, f'$  for the residue degrees of these extensions. Then ([Ser79, II, Cor. 1, p. 29])

$$ef = [L_w : K_v] = [L'_{w'} : K'_{v'}] = e'f'.$$

Consider the diagram of the residue extensions

$$\begin{array}{ccc} k(w) & \longrightarrow & k(w') \\ \uparrow & & \uparrow \\ k(v) & \longrightarrow & k(v'). \end{array}$$

In general, it is not true that  $k(w') = k(w) \otimes_{k(v)} k(v')$ .

We know that the bottom horizontal arrow is either an isomorphism or purely inseparable of degree  $p$ . If  $w|v$  is tamely ramified, then  $k(w)/k(v)$  is separable, so again we see that the horizontal arrows are either isomorphisms or both purely inseparable of degree  $p$  and  $k(w') = k(w) \otimes_{k(v)} k(v')$ . In either case  $f = f'$  and  $k(w')/k(v')$  is separable. From  $ef = e'f'$  it follows that  $(e', p) = 1$  and hence that  $w'|v'$  is tamely ramified.

Conversely, assume that  $w'|v'$  is tamely ramified. Let  $k^s \subseteq k(w)$  be the separable closure of  $k(v)$  in  $k(w)$ . By multiplicativity of the separable degree, we see that  $f' = [k^s : k(v)]$ . Thus  $f = f'p^\varepsilon$  where  $\varepsilon = 0$  or  $1$ . But by assumption  $(e', p) = 1$ , so from  $ef = e'f'$  it follows that  $\varepsilon = 0$ ,  $f = f'$  and  $e' = e$ . In particular,  $w|v$  is tamely ramified. ■

#### 4. NILPOTENT COVERINGS

In this section we prove a special case of [Theorem 1](#).

**Proposition 13.** *Let  $k$  be a field and  $X$  a regular, irreducible, separated, finite type  $k$ -scheme. There exists a good partial compactification  $(X, \overline{X}_{\text{nilp}})$  such that for every finite étale covering  $f : Y \rightarrow X$ , which can be dominated by a nilpotent Galois covering, the following are equivalent:*

- (a)  *$f$  is tame.*
- (b)  *$f$  is tamely ramified with respect to  $(X, \overline{X}_{\text{nilp}})$ .*

*Moreover, if  $X$  is quasi-projective, then we can choose  $\overline{X}_{\text{nilp}}$  to be quasi-projective as well.* □

The proof of this result follows an idea suggested by Moritz Kerz. Before we proceed with the argument, we recall the following main ingredient.

**Theorem 14** (“Inseparable local uniformization”, [\[Tem13, Cor. 1.3.3\]](#)). *Let  $X$  be an integral, separated finite type  $k$ -scheme. Then there exist morphisms  $\varphi_i : V_i \rightarrow X$ ,  $i = 1, \dots, r$ , with the following properties.*

- (a) *The  $V_i$  are regular, integral, separated, finite type  $k$ -schemes.*
- (b) *The maps  $V_i \rightarrow X$  are dominant and of finite type. They cover  $X$  in the following sense: Any valuation on  $k(X)$  (not necessarily discrete or rank 1) with center on  $X$  lifts to a valuation on some  $k(V_i)$  with center on  $V_i$ .*
- (c) *The induced extensions  $k(X) \subseteq k(V_i)$  are finite and purely inseparable,  $i = 1, \dots, r$ .* □

*Proof of Proposition 13.* Let  $\overline{X}$  be a normal compactification of  $X$  and apply Theorem 14 to  $\overline{X}$ . We obtain a map  $\varphi = \coprod_{i=1}^r \varphi_i : \coprod_{i=1}^r V_i \rightarrow \overline{X}$ , satisfying (a) – (c). Write  $U_i := \varphi_i^{-1}(X)$ . Then on each  $V_i$  there are finitely many codimension 1 points not contained in  $U_i$ . They correspond to geometric discrete valuations on  $k(V_i)$ , and hence give rise to a finite number of geometric discrete valuations  $v_1, \dots, v_n$  on  $k(X)$  (Proposition 6). It suffices to prove that a nilpotent Galois covering  $f : Y \rightarrow X$  is tame if and only if it is tamely ramified with respect to  $v_1, \dots, v_n$ , or equivalently, with respect to any good partial compactification  $(X, \overline{X}_{\text{nilp}})$  on which  $v_1, \dots, v_n$  have centers in codimension 1 points.

As  $X$  is regular, Remark 10 shows that  $f$  is tame if and only if it is tamely ramified with respect to all geometric discrete valuations of  $k(X)$ , in particular with respect to  $v_1, \dots, v_n$ .

Assume conversely that there is a geometric discrete valuation  $v$  on  $k(X)$  with respect to which  $f$  is wildly ramified. We want to show that  $f$  is wildly ramified with respect to one of the valuations  $v_1, \dots, v_n$ . As  $\overline{X}$  is proper,  $v$  has a center  $x$  on  $\overline{X}$  (possibly with  $\text{codim}_{\overline{X}}\{x\} > 1$ ). It follows that there exists a valuation  $w$ , on, say,  $k(V_1)$ , extending  $v$  and having a center on  $V_1$ . As  $k(V_1)/k(X)$  is purely inseparable,  $w$  is the only extension of  $v$  to  $k(V_1)$ .

Write  $f_1 : Y_1 \rightarrow U_1$  for the base change of  $f$  to  $U_1 = \varphi_1^{-1}(X)$ . According to Lemma 12,  $f_1$  is wildly ramified with respect to  $w$  and we claim that this implies that  $f_1$  is wildly ramified with respect to a codimension 1 point lying on  $V_1 \setminus U_1$ . Applying Lemma 12 again, this would imply that  $f$  is wildly ramified with respect to one of the valuations  $v_1, \dots, v_n$ , which is what we want to prove.

To prove the claim, we proceed along the lines of [Sch02, Prop. 1.10]. Let  $G = \text{Gal}(Y/X)$ . As  $G$  is nilpotent,  $G \cong P \times P'$  with  $P$  a  $p$ -group and  $P'$  a group of order prime to  $p := \text{char}(k)$ . In particular, the covering  $Y_1 \rightarrow U_1$  can be written as a tower of Galois coverings  $f_1 : Y_1 \xrightarrow{a} Y'_1 \xrightarrow{b} U_1$ , with  $\text{Gal}(b) = P$  and  $\text{Gal}(a) = P'$ . As the discrete valuation  $w$  of  $k(V_1)$  is wildly ramified in  $k(Y_1)$ , it follows that  $w$  is ramified in  $k(Y'_1)$ . On the other hand, as  $w$  has a center on the regular scheme  $V_1$ , the Zariski-Nagata purity theorem ([SGA1, X, Thm. 3.1]) implies that the normalization of  $V_1$  in  $k(Y'_1)$  is ramified over a closed subscheme of pure codimension 1 contained in  $V_1 \setminus U_1$ . As  $Y'_1 \rightarrow U_1$  has  $p$ -power degree, this ramification is wild. This proves the claim.

Finally, if  $X$  is quasi-projective, then we can choose  $\overline{X}$  to be a projective, normal compactification of  $X$ . The above construction applied to  $\overline{X}$  yields a quasi-projective  $\overline{X}_{\text{nilp}}$  with the desired properties. ■

## 5. APPLYING A THEOREM OF DRINFELD

We begin by establishing the following consequence of a Lefschetz theorem of Drinfeld.

**Proposition 15.** *Assume that the field  $k$  is either algebraically closed or finite. Let  $X'$  be a normal, projective, geometrically irreducible  $k$ -scheme*



and let  $X \subseteq X'$  be a smooth, dense open subscheme together with a geometric point  $\bar{x} \rightarrow X$ . Let  $\Sigma \subseteq X'$  be a closed subset satisfying the following conditions.

- (a)  $\text{codim}_{X'} \Sigma \geq 2$ ,
- (b)  $X' \setminus \Sigma$  is smooth,
- (c)  $(X' \setminus X) \setminus \Sigma$  is the support of a smooth divisor.

Then  $(X, X' \setminus \Sigma)$  is a good partial compactification and the profinite group  $\pi_1^{\text{tame}}((X, X' \setminus \Sigma), \bar{x})$  ([Definition 9](#)) is topologically finitely generated.

In particular, for a given  $N \in \mathbb{N}$ , the group  $\pi_1^{\text{tame}}((X, X' \setminus \Sigma), \bar{x})$  only has finitely many open subgroups of index  $\leq N$ .  $\square$

**Remark 16.** A profinite group which has only finitely many quotients of a given cardinality is called *small*. For a related, but smallness result for fundamental groups of varieties over finite fields, see [\[Hir15\]](#), which utilizes a deep finiteness theorem of Deligne ([\[EK12\]](#)).  $\square$

*Proof.* By [\[EK15, Prop. 6.2\]](#), which is a generalization of [\[Dri12, Appendix C\]](#), there exists a closed, smooth, irreducible  $k$ -curve  $C \subseteq X' \setminus \Sigma$  not contained in  $X' \setminus X$  and intersecting  $((X' \setminus \Sigma) \setminus X)_{\text{red}}$  transversely, such that for every finite irreducible étale covering  $Y \rightarrow X$  which is tamely ramified with respect to  $(X, X' \setminus \Sigma)$ , the pullback  $Y \times_X (C \cap X) \rightarrow (C \cap X)$  is irreducible. This means that for a geometric point  $\bar{c}$  of  $C \cap X$ , the induced map

$$\pi_1^{\text{tame}}(C \setminus X, \bar{c}) \rightarrow \pi_1^{\text{tame}}((X, X' \setminus \Sigma), \bar{c})$$

is surjective. The group on the left is known to be topologically finitely generated if  $k$  is algebraically closed or finite ([\[SGA1, XIII, Thm. 2.12\]](#) and [Lemma 11](#)).

Finally, according to [\[FJ08, Lemma 16.10.2\]](#), a topologically finitely generated group only has finitely many open subgroups of index  $\leq N$ .  $\blacksquare$

*Proof of Theorem 1.* Fix a geometric point  $\bar{x}$  of  $X$  and let  $X'$  be any projective normal compactification of  $X$ . We “approximate” the good partial compactification  $X_N$  in several steps. First, let  $X'_N$  be an open subset of  $X'$  containing  $X$ , such that  $(X, X'_N)$  is a good partial compactification and such that  $\text{codim}_{X'}(X'_N \setminus X) \geq 2$ . According to [Proposition 15](#), there are only finitely many open normal subgroups of index  $\leq N$  in  $\pi_1^{\text{tame}}((X, X'_N), \bar{x})$ . In particular, there are only finitely many Galois coverings  $Y_1/X, \dots, Y_m/X$ ,  $m \in \mathbb{N}$ , of degree  $\leq N$ , which are tamely ramified with respect to  $(X, X'_N)$  but not tame in the sense of [Definition 9, \(c\)](#). According to [Remark 10](#), for every  $i \in \{1, \dots, m\}$  we find a geometric discrete valuation  $v_i$  on  $k(X)$  with respect to which  $Y_i/X$  is wildly ramified. Blowing up  $X'$  repeatedly in the centers of the  $v_i$  and normalizing, we obtain a normal projective compactification  $X''$  of  $X$ , such that each  $v_i$  is centered in a codimension 1 point of  $X''$  (see [Proposition 8](#)). Let  $X''_N$  be an open subset of  $X''$  containing  $X$  and all codimension 1 points of  $X''$ , such that  $(X, X''_N)$  is a good partial compactification of  $X$ . It follows that a Galois covering of degree  $\leq N$  of  $X$  is tame if and only if it is tamely ramified with respect to the good partial compactification  $(X, X''_N)$ .



Now let  $f : Y \rightarrow X$  be a Galois étale covering with group  $G$  such that  $G$  contains a normal nilpotent subgroup  $H$  of index  $\leq N$ . It factors as

$$\begin{array}{ccc}
 Y & & \\
 \downarrow f & \searrow f_H & \\
 & Y' & \\
 & \swarrow f_{G/H} & \\
 X & & 
 \end{array} \tag{2}$$

where  $f_H$  is Galois étale with group  $H$  and  $f_{G/H}$  is Galois étale with group  $G/H$ ; in particular,  $\deg(f_{G/H}) \leq N$ . If  $f$  is tamely ramified with respect to  $(X, X''_N)$ , then so is  $f_{G/H}$ , and using [Proposition 15](#) again, we see that there are only finitely many possibilities for  $f_{G/H}$ ; we write them as  $Y'_1/X, \dots, Y'_{m'}/X$ .

For  $i = 1, \dots, m'$ , we apply [Proposition 13](#) to the  $k$ -scheme  $Y'_i$  and obtain a finite set  $\{w_{ij}\}_{j=1, \dots, r_i}$  of geometric discrete valuations on  $k(Y'_i)$  with the property that a nilpotent covering of  $Y'_i$  is tame if and only if it is tamely ramified with respect to  $w_{ij}$ ,  $j = 1, \dots, r_i$ . As  $Y'_i/X$  is finite, [Proposition 6](#) shows that the restriction  $v_{ij} := (w_{ij})|_{k(X)}$  is a geometric discrete valuation on  $k(X)$ .

Now assume that  $f$  is tamely ramified with respect to  $(X, X''_N)$  and with respect to all  $v_{ij}$ . As  $\deg(f_{G/H}) \leq N$ ,  $f_{G/H}$  is tame by construction of  $X''_N$ , and  $f_{G/H}$  is one of the coverings  $Y'_1/X, \dots, Y'_{m'}/X$ . Moreover, as the valuations  $w_{ij}$  arise from [Proposition 13](#), it follows that if  $f_H$  is wildly ramified, then there exists a pair  $(i, j)$ , such that  $f_H$  is wildly ramified with respect to  $w_{ij}$ , which implies that  $f$  is wildly ramified with respect to  $v_{ij}$ ; contradiction. Thus  $f$  is tame.

Finally, blow up  $X''$  repeatedly in the centers of the  $v_{ij}$  and normalize to obtain  $X'''$ , a normal projective compactification of  $X$ , such that each  $v_{ij}$  is centered in a codimension 1 point of  $X'''$ . Let  $X_N$  be a suitable open subset of  $X'''$  containing  $X$  and all the centers of the  $v_{ij}$ . We proved that  $f$  as in (2) is tame, if it is tamely ramified with respect to  $(X, X_N)$ . The converse is true by definition. ■

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